

Tutorial 9

Cooperative games

Let $\mathcal{A} = \{A_1, \dots, A_n\}$ be the set of players. Let X_i be the set of strategies of Player A_i ($i = 1, \dots, n$). In a cooperative game, the players can cooperate by forming coalitions. Players in the same coalition can share their utility.

Coalition. We call any subset $S \subset \mathcal{A}$ a coalition.

If the strategies and the corresponding payoff vectors are given, we define $\nu(S)$ to be the total maximin value of S when the game is viewed as a two-person non-cooperative game between S and S^c . More precisely,

$$\nu(S) = \text{the value of the payoff matrix of } S.$$

For convenience, we write $\nu(A) = \nu(\{A\})$. As a function on $2^{\mathcal{A}}$ (the power set of \mathcal{A}), ν satisfies

$$\nu(S \cup T) \geq \nu(S) + \nu(T), \text{ for any } S, T \subset \mathcal{A} \text{ with } S \cap T = \emptyset.$$

Characteristic function (form). A function $\nu : 2^{\mathcal{A}} \rightarrow \mathbb{R}^+$ is called a characteristic function (form) if $\nu(\emptyset) = 0$ and

$$\nu(S \cup T) \geq \nu(S) + \nu(T), \text{ for any } S, T \subset \mathcal{A} \text{ with } S \cap T = \emptyset.$$

Essential game: $\nu(\mathcal{A}) > \sum_{i=1}^n \nu(A_i)$.

Inessential game: $\nu(\mathcal{A}) = \sum_{i=1}^n \nu(A_i)$.

Solving a cooperative game: find reasonable ways to split the total payoff $\nu(\mathcal{A})$ among the players.

In this sense, only essential games are of interest, since we have

Theorem 1. *If ν is inessential, then*

$$\nu(S) = \sum_{A \in S} \nu(A), \text{ for any } S \subset \mathcal{A}.$$

Imputation.

A vector $(x_1, \dots, x_n) \in \mathbb{R}^n$ is called an imputation for ν if

(i) $\nu(A_i) \leq x_i$ for $1 \leq i \leq n$.

(ii) $\sum_{i=1}^n x_i = \nu(\mathcal{A})$.

Let $I(\nu)$ to be the set of imputations. We understand $I(\nu)$ as the set of all possible ways to split the total payoff.

Solution concept 1: the core.

The *core* of ν is defined by

$$C(\nu) = \{\mathbf{x} \in I(\nu) : \mathbf{x} \not\prec_S \mathbf{y} \text{ for any } \mathbf{y} \in I(\nu) \text{ and any } S \subset \mathcal{A}\}.$$

where $\mathbf{x} \prec_S \mathbf{y}$ means $x_i < y_i$ for $A_i \in S$ and $\sum_{A_i \in S} y_i \leq \nu(S)$.

Characterization of $C(\nu)$.

Theorem 2. *Let $\mathbf{x} = (x_1, \dots, x_n) \in I(\nu)$. Then $\mathbf{x} \in C(\nu)$ if and only if*

$$\nu(S) \leq \sum_{A_i \in S} x_i \text{ for any } S \subset \mathcal{A}.$$

Disadvantage: $C(\nu)$ may be an empty set.

Solution concept 2: Shapley values.

For each player A_i , define the Shapley value of A_i by

$$\phi_i = \sum_{A_i \in S \subset \mathcal{A}} \frac{(n - |S|)! (|S| - 1)!}{n!} (\nu(S) - \nu(S \setminus \{A_i\})).$$

Let $\phi(v) = (\phi_1, \dots, \phi_n)$.

ϕ_i can be viewed as the average contribution of Player A_i .

Exercise 1. Let $\mathcal{A} = \{1, \dots, N\}$. Verify that for each $i \in \mathcal{A}$,

$$\sum_{i \in S \subset \mathcal{A}} \frac{(N - |S|)! (|S| - 1)!}{N!} = 1.$$

Solution.

$$\begin{aligned} \sum_{i \in S \subset \mathcal{A}} (N - |S|)! (|S| - 1)! &= \sum_{k=1}^N \sum_{\substack{i \in S \subset \mathcal{A} \\ |S|=k}} (N - k)! (k - 1)! \\ &= \sum_{k=1}^N \binom{N-1}{k-1} (N - k)! (k - 1)! \\ &= \sum_{k=1}^N (N - 1)! = N!. \end{aligned}$$

Advantages of Shapley values:

- (i) $\phi(\nu) \in I(\nu)$.
- (ii) $\phi(\nu)$ is the unique payoff allocation which satisfies the axioms for Shapley value.
- (iii) If the game is convex, then $\phi(\nu) \in C(\nu)$, in particular $C(\nu) \neq \emptyset$.

Convex games. We say a game ν is convex if

$$\nu(S \cup T) \geq \nu(S) + \nu(T) - \nu(S \cap T)$$

for any $S, T \subset \mathcal{A}$.

Exercise 2. Consider a 3-person game with player set $\mathcal{A} = \{1, 2, 3\}$ and each of the players has strategy set $\{0, 1\}$. Suppose the payoffs are given in Table 1.

Strategies	Payoff vectors
(0, 0, 0)	(2, 3, 4)
(0, 0, 1)	(4, 6, 4)
(0, 1, 0)	(7, 4, 2)
(0, 1, 1)	(3, 2, 9)
(1, 0, 0)	(4, 3, 5)
(1, 0, 1)	(5, 8, 7)
(1, 1, 0)	(3, 1, 5)
(1, 1, 1)	(3, 6, 5)

Table 1

{2,3} \ {1}	{1}	0	1
	(0, 0)	(7, 2)	(8, 4)
(0, 1)	(10, 4)	(12, 5)	
(1, 0)	(6, 7)	(6, 3)	
(1, 1)	(11, 3)	(11, 3)	

Table 2

(i) Find the characteristic function ν .

(ii) Find the core and draw the region on the x_1 - x_2 plane.

(iii) Find the Shapley values.

Solution. (i) To find $\nu(\{1\})$, we consider the two-person non-cooperative game between coalitions $\{1\}$ and $\{2, 3\}$. The bimatrix is given by Table 2.

Since

$$A = \begin{pmatrix} 7 & 8 \\ 10 & 12 \\ 6 & 6 \\ 11 & 11 \end{pmatrix}, B = \begin{pmatrix} 2 & 4 \\ 4 & 5 \\ 7 & 3 \\ 3 & 3 \end{pmatrix}, B^T = \begin{pmatrix} 2 & 4 & 7 & 3 \\ 4 & 5 & 3 & 3 \end{pmatrix}.$$

We have $\nu(\{2, 3\}) = \nu_A = 11$ and $\nu(\{1\}) = \nu_{B^T} = 3$. By the same method,

we have

$$\begin{aligned}
 \nu(\{1\}) &= 3 \\
 \nu(\{2\}) &= 3 \\
 \nu(\{3\}) &= 4 \\
 \nu(\{1, 2\}) &= 9 \\
 \nu(\{1, 3\}) &= 10 \\
 \nu(\{2, 3\}) &= 11 \\
 \nu(\{1, 2, 3\}) &= 20.
 \end{aligned}$$

(ii) Let $(x_1, x_2, x_3) \in C(\nu)$. Then by the characterization of the core, we have

$$\left\{ \begin{array}{l}
 x_1 \geq 3 \\
 x_2 \geq 3 \\
 x_3 \geq 4 \\
 x_1 + x_2 \geq 9 \\
 x_1 + x_3 \geq 10 \\
 x_2 + x_3 \geq 11 \\
 x_1 + x_2 + x_3 = 20.
 \end{array} \right.$$

Equivalently,

$$\left\{ \begin{array}{l}
 3 \leq x_1 \leq 9 \\
 3 \leq x_2 \leq 10 \\
 9 \leq x_1 + x_2 \leq 16 \\
 x_1 + x_2 + x_3 = 20.
 \end{array} \right.$$

Hence the region of $C(\nu)$ in x_1 - x_2 plane is shown in Figure 1.

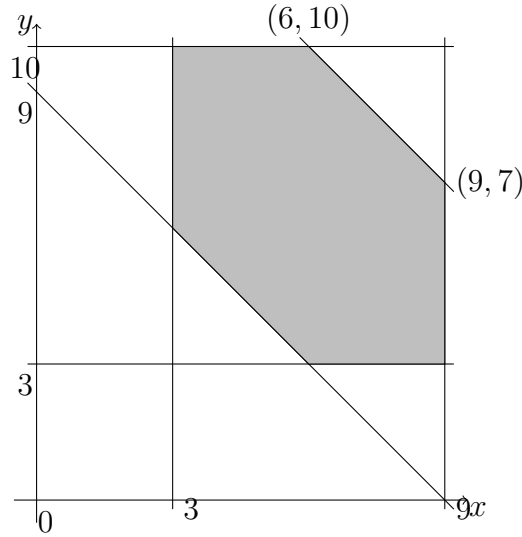


Figure 1

(iii) Then Shapley value of the Player 1 is

$$\begin{aligned} \phi_1 = \frac{1}{3!} & [(3-1)!(1-1)! \times 3 + (3-2)!(2-1)! \times (9-3) + (3-2)!(2-1)! \times (10-4) \\ & + (3-3)!(3-1)! \times (20-11)] = 6. \end{aligned}$$

Similarly, we have $\phi_2 = \frac{13}{2}$, $\phi_3 = \frac{15}{2}$.

Exercise 3. In a 3-person cooperative game, the three players together can obtain 1\$, any two players can obtain α , and a single player can obtain zero. Describe the core of the game.

Solution. Assume that the player set is $\mathcal{A} = \{1, 2, 3\}$. Then we have $\nu(\mathcal{A}) = 1$, $\nu(\{1\}) = \nu(\{2\}) = \nu(\{3\}) = 0$ and $\nu(\{1, 2\}) = \nu(\{1, 3\}) = \nu(\{2, 3\}) = \alpha$.

Assume that $(x_1, x_2, x_3) \in C(\nu)$. Then

$$\begin{cases} x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, & (1a) \\ x_1 + x_2 \geq \alpha, x_1 + x_3 \geq \alpha, x_2 + x_3 \geq \alpha, & (1b) \\ x_1 + x_2 + x_3 = 1. & (1c) \end{cases}$$

By (1b), we have $x_1 + x_2 + x_3 \geq \frac{3\alpha}{2}$. Then by (1c) we have

(1) $C(\nu) = \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$ if $\alpha = \frac{2}{3}$;

(2) $C(\nu) = \emptyset$ if $\alpha > \frac{2}{3}$;

(3) $C(\nu) = \{(x_1, x_2, x_3) : (x_1, x_2, x_3) \text{ satisfies (1a), (1b) and (1c)}\}$ if $\alpha < \frac{2}{3}$,

in which case $C(\nu)$ has a continuum of points.